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State-Variable Equivalents to Continuous and Pulse Transfer Functions

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Introduction

IN the state-variable formulation a first-order differential or difference equation is written for each degree of freedom of the system under study. State-variable equations are the language of modern control theory, including optimization and estimation. Papers in recent issues of this journal have used state-variable equations in analyses of nonlinear missile guidance systems, aircraft lateral handling qualities, adaptive flight control systems, aerodynamic parameter estimation, and digital autopilot response.

This Note is intended to facilitate the transformation into state-variable form for that portion of complete guidance and control systems that consists of isolated transfer functions. These transfer functions typically include shaping networks, instruments, and actuators. Tables are presented of transfer function-state variable equivalents.

Continuous Transfer Functions

Continuous one-input one-output transfer functions of the Laplace transform variable s have the form

$$\frac{y(s)}{m(s)} = \frac{y_p s^p + y_{p-1} s^{p-1} + \cdots + y_0}{m_n s^n + m_{n-1} s^{n-1} + \cdots + m_0} \tag{1}$$

Table 1 State-variable forms for transfer-function denominators

No.	Denominators $m(s)$	State equations
D1	$s+d$	$\dot{x}_1 = -dx_1 + m$
D2	$(s+d_1)(s+d_2)$	$\dot{x}_1 = x_2$
		$\dot{x}_2 = -d_1 d_2 x_1 - (d_1+d_2)x_2 + m$
D3	$s^2 + d_1 s + d_2$	$\dot{x}_1 = x_2$
		$\dot{x}_2 = -d_2 x_1 - d_1 x_2 + m$
D4	$(s+d_1)(s+d_2)(s+d_3)$	$\dot{x}_1 = x_2$
		$\dot{x}_2 = x_3$
		$\dot{x}_3 = -d_1 d_2 d_3 x_1 - (d_1 d_2 + d_2 d_3 + d_3 d_1)x_2 - (d_1 + d_2 + d_3)x_3 + m$
D5	$(s+d_1)(s^2 + d_2 s + d_3)$	$\dot{x}_1 = x_2$
		$\dot{x}_2 = x_3$
		$\dot{x}_3 = -d_1 d_3 x_1 - (d_1 d_2 + d_3)x_2 - (d_1 + d_2)x_3 + m$

Table 2 State-variable output equations

No.	Transfer functions $y(s)/m(s)$	Output equations
1.1	$K/D1$	$y = Kx_1$
1.2	$K(s+n)/D1$	$y = K[(n-d)x_1 + m]$
2.1	$K/D2$	$y = Kx_1$
2.2	$K(s+n)/D2$	$y = K(nx_1 + x_2)$
2.3	$K(s+n_1)(s+n_2)/D2$	$y = K[(n_1 n_2 - d_1 d_2)x_1 + (n_1 + n_2 - d_1 - d_2)x_2 + m]$
2.4	$K(s^2 + n_1 s + n_2)/D2$	$y = K[(n_2 - d_1 d_2)x_1 + (n_1 - d_1 - d_2)x_2 + m]$
3.1	$K/D3$	$y = Kx_1$
3.2	$K(s+n)/D3$	$y = K(nx_1 + x_2)$
3.3	$K(s+n_1)(s+n_2)/D3$	$y = K[(n_1 n_2 - d_2)x_1 + (n_1 + n_2 - d_1)x_2 + m]$
3.4	$K(s^2 + n_1 s + n_2)/D3$	$y = K[(n_2 - d_2)x_1 + (n_1 - d_1)x_2 + m]$
4.1	$K/D4$	$y = Kx_1$
4.2	$K(s+n)/D4$	$y = K(nx_1 + x_2)$
4.3	$K(s+n_1)(s+n_2))/D4$	$y = K[n_1 n_2 x_1 + (n_1 + n_2)x_2 + x_3]$
4.4	$K(s^2 + n_1 s + n_2)/D4$	$y = K(n_2 x_1 + n_1 x_2 + x_3)$
4.5	$K(s+n_1)(s+n_2)(s+n_3)/D4$	$y = K[(n_1 n_2 n_3 - d_1 d_2 d_3)x_1 + (n_1 n_2 + n_2 n_3 + n_3 n_1 - d_1 d_2 - d_2 d_3 - d_3 d_1)x_2 + (n_1 + n_2 + n_3 - d_1 - d_2 - d_3)x_3 + m]$
4.6	$K(s+n_1)(s^2 + n_2 s + n_3)/D4$	$y = K[(n_1 n_3 - d_1 d_2 d_3)x_1 + (n_1 n_2 + n_3 - d_1 d_2 - d_2 d_3 - d_3 d_1)x_2 + (n_1 + n_2 - d_1 - d_2 - d_3)x_3 + m]$
5.1	$K/D5$	$y = Kx_1$
5.2	$K(s+n)/D5$	$y = K(nx_1 + x_2)$
5.3	$K(s+n_1)(s+n_2)/D5$	$y = K[n_1 n_2 x_1 + (n_1 + n_2)x_2 + x_3]$
5.4	$K(s^2 + n_1 s + n_2)/D5$	$y = K(n_2 x_1 + n_1 x_2 + x_3)$
5.5	$K(s+n_1)(s+n_2)(s+n_3)/D5$	$y = K[(n_1 n_2 n_3 - d_1 d_3)x_1 + (n_1 n_2 + n_2 n_3 + n_3 n_1 - d_1 d_2 - d_3)x_2 + (n_1 + n_2 + n_3 - d_1 - d_2)x_3 + m]$
5.6	$K(s+n_1)(s^2 + n_2 s + n_3)/D5$	$y = K[(n_1 n_3 - d_1 d_3)x_1 + (n_1 n_2 + n_3 - d_1 d_2 - d_3)x_2 + (n_1 + n_2 - d_1 - d_2)x_3 + m]$

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An equivalent state-variable form given by Tou[1] is the linear differential matrix equation with constant coefficients

$$\dot{x}(t) = Ax(t) + Bm(t) \tag{2a}$$

$$y(t) = Cx(t) + Dm(t) \tag{2b}$$

Equation (2a) is the state equation, in which x is a column matrix of the same order n as the degree of the denominator polynomial $m(s)$. Equation (2b) is the output matrix (non-differential) equation, in which C is an output matrix and D is a transmission matrix. D is nonzero in the state-variable forms that follow only when the degree of the numerator p is equal to the degree of the denominator n in the transfer function Eq. (1).

The transformation from Eq. (1) to Eq. (2) is not unique, and Tou[1] provides three alternate transformation methods, called direct, parallel, and iterative programming. Each method yields different definitions for the matrices A to D, in the general case. However, there appears to be no practical advantage in simulation or synthesis for one state-variable form over another, and a single transformation from Eq. (1) to Eq. (2) is used in this Note. This is the "direct programming" method.[1]

The numerator and denominator functions $y(s)$ and $m(s)$ are treated independently. Table 1 provides the direct-programming equivalents to the denominators, in the form of the state equation (2a). Table 2 provides the output equation (2b), which is a function of both numerators and denominators. The use of these tables is illustrated by the following example. For the transfer function

$$\frac{y}{m}(s) = \frac{K}{s(s+10)(s+20)} \tag{3}$$

the denominator D4 of Table 1 and the function 4.1 of Table 2 apply, giving the state-variable equations

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3$$
$$\dot{x}_3 = -200x_2 - 30x_3 + m \qquad y = kx_1 \tag{4}$$

Pulse-Transfer Functions

The block diagram operations involved in Tou's direct-programming method for continuous functions are the same for pulse-transfer functions when the following substitutions are made, with k and $k+1$ sampling time indices:

$$s \to z$$
$$x_1 \to (x_1)_k \qquad \dot{x}_1 \to (x_1)_{k+1}$$
$$x_2 \to (x_2)_k \qquad \dot{x}_2 \to (x_2)_{k+1}, \quad \text{etc.} \tag{5}$$

Thus, Tables 1 and 2 apply equally well to reduction of pulse-transfer functions to difference equations in state-variable form. To illustrate, take the pulse-transfer function

$$\frac{Y}{m}(z) = \frac{1 + az^{-1} + bz^{-2}}{1 + cz^{-1} + dz^{-2}} = \frac{z^2 + az + b}{z^2 + cz + d} \tag{6}$$

The D3 denominator of Table 1 and the function 3.4 of Table 2 apply, giving the state-variable difference equations

$$(x_1)_{k+1} = (x_2)_k$$
$$(x_2)_{k+1} = -d(x_1)_k - c(x_2)_k + (m)_k$$
$$(y)_k = (b-d)(x_1)_k + (a-c)(x_2)_k + (m)_k \tag{7}$$

Reference

[1] Tou, J.T., *Modern Control Theory,* 1st Ed., McGraw-Hill, New York, 1964, pp. 69-72.